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A SIMPLICIAL GROUP CONSTRUCTION FOR BALANCED PRODUCTS

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§1. INTRODUCTION

LET X be a right G-space, where G is a discrete group. In this paper, we construct a simplicial group $J_G(X)$ whose homotopy type is that of $\Omega C(X)$, where $C(X)$ is the mapping cone of the inclusion of a fibre $X = EG \underset{G}{\times} X$. There is also a based version, whose homotopy type is $\Omega\bar{C}(X)$, where $\bar{C}(X)$ is the mapping cone of the inclusion $X \to EG \underset{G}{\propto} X$ of a fibre. (In both cases, EG is a contractible space on which G acts freely). This simplicial group is usually smaller than the free loop group construction on $C(X)$ or $\bar{C}(X)$, and one expects that studying it will yield interesting information about the homotopy groups of $C(X)$ or $\bar{C}(X)$. The spaces which occur as $\bar{C}(X)$ are of interest in homotopy theory; for instance, if ρ is a representation of a finite group and $S(\rho)$ denotes its one point compactification, then the mapping cone $S(\rho) \to EG \underset{G}{\times} S(\rho)$ is the quotient of the Thom complex of the vector bundle $EG \underset{G}{\times} \rho$ over BG by its lowest dimensional cell. Thus, all stunted projective spaces occur. Filtering these groups by the mod p lower central series (as in [2]) should produce interesting Adams-type spectral sequences for $\pi_*(C(X))$, or should at least shed light on the usual Adams spectral sequence. In the case when X is a trivial based G-space, the homotopy type is $\Omega(BG \wedge X)$. When $G = \mathbb{Z}$, this specializes to the usual James construction[3, 5].

We describe the construction when $G = \mathbb{Z}/2$. Let X be a simplicial G-set; let T be the non-trivial element of G. Then $J_G(X)$ is obtained in each dimension n as the quotient of the free group on X_n by the relations $x \cdot Tx = e$. If we have a based G-space we also impose the relation $* = e$. The description points out one of the defects of the construction, namely that J_G does not always produce free groups. For instance, for trivial G-sets, J_G produces groups which are free products of the group G.

The paper is brief; the statements of the results and their proofs are in §2. We work simplicially. For a discussion of simplicial topology, see [4]; and for the properties of bisimplicial sets, see [1]. The author wishes to express his thanks to the referee, who both simplified and generalized the results in an earlier draft of this paper. Without his comments, only a very special and rudimentary version of the results we prove would be available.

§2. THE CONSTRUCTION J_G; PROOFS OF THE THEOREMS

Let G be a group. Throughout, F denotes the free group fun from sets to groups.

Definition 1. Let X be a right G-set. Then we define the group $J_G(X)$ to be $F(X \times G)/N$ where N denotes the normal subgroup generated by the elements of the form $[x, gh]^{-1}[x, g][xg, h]$. ($[x, g]$ denotes the pair $(x, g) \in X \times G \subseteq F(X \times G)$). If X is a based right G-set, with basepoint $*$, then $J_G(X, *) = J_G(X)/M$, where M is the normal subgroup generated by the elements $[*, g]$.

Thus, $J_G(X)$ is obtained by requiring the relations $[x, g][xg, h] = [x, gh]$ to hold. If X_*

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is a simplicial right G-set, we apply J_G in each dimension and obtain a simplicial group $J_G(X_*)$. It is the homotopy type of $J_G(X_*)$ we wish to analyze. In order to study $J_G(X_*)$, we must first discuss $J_G(H\backslash G)$ where $H\backslash G$ denotes the right G-set of right cosets of a subgroup $H \subseteq G$. Let $x \in H\backslash G$ denote the coset H, and let $\overline{H\backslash G} = H\backslash G - \{x\}$.

LEMMA 2. *$J_G(H\backslash G) \cong H * F(\overline{H\backslash G})$. If $\bar{g}_1, \ldots, \bar{g}_k$ are coset representatives for the elements of $\overline{H\backslash G}$, a basis for the free factor is given by $[x, \bar{g}_1], \ldots, [x, \bar{g}_k]$.*

Proof. We first form the group $\bar{J}$, which is the free group on symbols $[g]$, $g \in G$, modulo the relations $\{[h][g] = [hg],\ h \in H,\ g \in G\}$. We define a map $\varphi\colon \bar{J} \to J_G(H\backslash G)$ by $[g] \to [x, g]$. This is clearly well-defined, and we claim that φ is an isomorphism.

(a) φ is subjective. For, let $[xg_1, g_2]$ be any generator of $J_G(H\backslash G)$. Then by the relations defining J_G, $[xg_1, g_2] = [x, g_1]^{-1}[x, g_1g_2] = \varphi([g_1]^{-1}[g_1g_2])$.

(b) φ is injective. Let $\bar{g}_1, \ldots, \bar{g}_k$ be as in the statement of the lemma. We define a map $S\colon F(H\backslash G \times G) \to \bar{J}$ *by*

$$S([x, g]) = [g]$$

$$S([x\bar{g}_j, g]) = [\bar{g}_j]^{-1}[\bar{g}_j g].$$

Of course, any element in $\overline{H\backslash G}$ may be written as uniquely as $x\bar{g}_j$, so this is a well-defined map. We now show that S is in fact well-defined on the quotient $F(H\backslash G \times G)/N = J_G(H\backslash G)$. We must therefore show that $S([x\bar{g}_i, g][x\bar{g}_i g, g']) = S([x\bar{g}_i, gg'])$. We suppose first that $\bar{g}_i g \notin H$, and write $\bar{g}_i g = h\bar{g}_j$, $h \in H$. This uniquely determines h and $\bar{g}_j$. Now,

$$\begin{aligned} S([x\bar{g}_i, g][x\bar{g}_i g, g']) &= [\bar{g}_i]^{-1}[\bar{g}_i g] S([xh\bar{g}_j, g']) \\ &= [\bar{g}_i]^{-1}(h\bar{g}_j) \cdot S([x\bar{g}_j, g']) = [\bar{g}_i]^{-1}[h][\bar{g}_j][\bar{g}_j]^{-1}[\bar{g}_j g'] \\ &= [\bar{g}_i]^{-1}[h][\bar{g}_j g'] = [\bar{g}_i]^{-1}[h\bar{g}_j g'] = [\bar{g}_i]^{-1}[\bar{g}_i g g'] \\ &= S([x\bar{g}_i, gg']). \text{ If } \bar{g}_i g \in H, \end{aligned}$$

$$S([x\bar{g}_i, g][x\bar{g}_i g, g']) = [\bar{g}_i]^{-1}[\bar{g}_i g][g'] = [\bar{g}_i]^{-1}[\bar{g}_i g g'] = S([x\bar{g}_i, gg']).$$

The relations involving $[x, g]$ are immediate. Thus, we have constructed $S\colon J_G(H\backslash G) \to \bar{J}$; it is an easy calculation that $S\varphi$ and φS are the respective identity maps, so φ is an isomorphism.

We now define a map $q\colon H * F(\overline{H\backslash G}) \to \bar{J}$ by $h \to [h]$, $h \in H$, and $[x\bar{g}_i] \to [\bar{g}_i]$. q is clearly surjective, since $[h][\bar{g}_i] = [g]$, if h, $\bar{g}_i$ are chosen so that $h\bar{g}_i = g$. We form a section $r\colon \bar{J} \to H * F(\overline{H\backslash G})$ by setting $r([g]) = h \cdot [x\bar{g}_i]$, where $g = h\bar{g}_i$, $h \in H$. We'll need to show that r is well-defined, i.e. that $r([h][g]) = r([hg])$, $h \in H$, $g \in G$. But this is clear, since $r([h][g]) = h \cdot h'[\bar{g}_i] = r([hh'\bar{g}_i]) = r([hg])$, where $g = h'\bar{g}_i$, $h' \in H$. Again, gr and rg are the respective identity maps, which proves that $J_G(H\backslash G) \cong H * F(\overline{H\backslash G})$. The statement about the basis for the free factor is clear, since $\varphi(q([x\bar{g}_i])) = \varphi([\bar{g}_i]) = [x, \bar{g}_i]$. ■

We now recall the simplicial bar construction for a group G (see [4]). $W(G)$ is a free, contractible, simplicial left G-set. It is defined in each dimension by $W(G)_n = G^{n+1}$. Elements of $W(G)_n$ are written $[g_0|g_1| \ldots |g_n]$, $g_i \in G$. The face and degeneracy maps are given by

$$d_i[g_0|g_1| \ldots |g_n] = [g_0|g_1| \ldots |g_{i-1}|g_i g_{i+1}|g_{i+2}| \ldots |g_n], \quad 0 \le i < n$$

$$d_n[g_0|g_1| \ldots |g_n] = [g_0|g_1| \ldots |g_{n-1}]$$

$$S_i[g_0|g_1| \ldots |g_n] = [g_0|g_1| \ldots |g_i|e|g_{i+1}| \ldots |g_n].$$

We define the left G-action on $W(G)_n$ by $g \cdot [g_0|\dots|g_n] = [gg_0|g_1|\dots|g_n]$. Given a right G-set X, we define $X \times_G W_n(G)$ to be the orbit space of $X \times W_n(G)$ under the left G-action defined by $g \cdot (x, w) = (xg^{-1}, gw)$, $x \in X$, $w \in W_n(G)$. $X \times_G W(G)$ is a simplicial set, with face and degeneracy maps induced from those on $W(G)$. We then define a natural transformation (natural in X)

$$\theta_n\colon X \times_G W_n(G) \to W_n(J_G(X))/J_G(X)$$

by the equation $\theta_n(x, [g_0|g_1|\dots|g_n]) = [[x, g_0]|[xg_0, g_1]|\dots|[xg_0\dots g_{n-1}, g_n]]$. We must verify that this is well-defined; thus we compute.

$$\theta_n(xg^{-1}, [gg_0|g_1|\dots|g_n]) = [[xg^{-1}, gg_0]|[xg_0, g_1]|[xg_0g_1, g_2]|\dots|[xg_0\dots g_{n-1}, g_n]]$$

In $J_G(X)$, we have the relation $[xg^{-1}, gg_0] = [xg^{-1}, g][x, g_0]$, so we obtain the equation $\theta_n(xg^{-1}, gw) = [xg^{-1}, g]\theta_n(x, w)$, which shows that θ_n is well-defined. One also easily checks the identities $d_i\theta_n = \theta_{n-1}d_i$, $s_i\theta_n = \theta_{n+1}s_i$, which show that the θ_n's together give a simplicial map

$$\theta\colon X \times_G W(G) \to W(J_G(X))/J_G(X) = BJ_G(X).$$

Here, $BJ_G(X)$ is the classifying space for $J_G(X)$ (see [4]); it is a $K(J_G(X), 1)$-space. Now, note that there is a natural inclusion i of X, viewed as a discrete simplicial set, into $X \times_G W(G)$. It is defined in dimension zero by $x \to x \times [e]$, and this determines the simplicial map in all dimensions. We define a simplicial set $C(X) = X \times_G W(G)/i(X)$, i.e. we identify all the elements of $i(X)$ to one point.

Lemma 3. *$\theta \circ i$ is the constant map sending X to the basepoint in $BJ_G(X)$.*

Proof. A typical element in $i(X)$ is of the form $(x, [e|e|\dots|e])$; $\theta(x, [e|\dots|e]) = [[x, e]|\dots|[x, e]]$. The relation $[x, e]\cdot[x, e] = [x, e]$ in $J_G(X)$ shows that $[x, e]$ is the identity element in $J_G(X)$. Therefore, $\theta(x, [e|\dots|e]) = [e|\dots|e]$, which is a degeneracy operator applied to the basepoint in $BJ_G(X)$. ■

Corollary 4. *There is a factorization*

$$\begin{tikzcd} X \times_G \bar{W}(G) \arrow[r, "\theta"] \arrow[d] & BJ_G(X) \\ C(X) \arrow[ur, "\bar{\theta}"'] & \end{tikzcd}$$

Theorem 5. *The map $\bar{\theta}\colon C(X) \to BJ_G(X)$ is a weak homotopy equivalence. $|\bar{\theta}|$ is a homotopy equivalence.*

Proof. We note first that if X_1 and X_2 are the two right G-sets, then $C(X_1 \mathring{\cup} X_2) = C(X_1) \vee C(X_2)$. Also, $J_G(X_1 \mathring{\cup} X_2) = J_G(X_1) * J_G(X_2)$, where $*$ denotes free product. It is a standard result that $B(J_G(X_1) * J_G(X_2)) \cong BJ_G(X_1) \vee BJ_G(X_2)$, and that these identifications are natural, so it will suffice to prove the result for the right coset space $H\backslash G$, $H \subseteq G$. We will verify that both spaces are $K(\pi, 1)$'s; this is obvious for $BJ_G(H\backslash G)$.

For $C(X)$, we note that $C(X)$ has the homotopy type of the mapping cone of the map $H\backslash G \to H\backslash G \underset{G}{\times} \bar{W}(G) \cong BH$, in which $H\backslash G$ is sent to $H\backslash G \times [e]$. Since $H\backslash G \underset{G}{\times} \bar{W}(G)$ is connected, this mapping cone has the homotopy type of $BH \vee \Sigma(\overline{H\backslash G})$. Here, Σ denotes unreduced suspension. $\Sigma(\overline{H\backslash G})$ is a wedge of circles, so $C(X)$ is a $K(\pi, 1)$, where $\pi = H * F$, and F is a free group on generators corresponding to the elements of $\overline{H\backslash G}$. We therefore need only verify that $\bar{\theta}$ induces an isomorphism on π_1. The elements of $\pi_1(H\backslash G \underset{G}{\times} \bar{W}(G)) \cong H$ are given by the one-simplices $(x, [e|h])$, where x is the coset H, and h is an element of H. In $C(H\backslash G)$, we have the additional generators $(x, [\bar{g}_i|e])$, where the $\bar{g}_i$'s are coset representatives as in the proof of Lemma 2, which generate the free factor. One now readily checks, via Lemma 2, that these generators map to corresponding generators in $J_G(H\backslash G) \cong \pi_1(BJ_G(H\backslash G))$, and hence the isomorphism statement for π_1, which proves the theorem. ■

Let B denote the classifying space functor from groups to simplicial sets; $BG = W(G)/G$. If we consider a simplicial right G-set X, and apply J_G to it, we obtain the simplicial group $J_G(X)$. If we apply B to $J_G(X)$, we obtain a bisimplicial set, which we write $BJ_G(X)_{**}$ (see [1]). We may also consider the bisimplicicial set $X \underset{G}{\times} W(G)$, which in bidimension (m, n) is $X_m \underset{G}{\times} W(G)_n$, and we may identify the sub-bisimplicial set whose bisimplices in bidimension (m, n) are $X_m \underset{G}{\times} [e|\ldots|e]$ to a point, obtaining a bisimplicial set $\mathscr{C}_{**}(X)$. One notes that the maps θ_n defined above yield a map of bisimplical sets Θ: $\mathscr{C}_{**}(X) \to BJ_G(X)_{**}$. Let $|\ |$ denote geometric realization, both for simplicial and bisimplicial sets.

PROPOSITION 6. *The inducedmap* $|\Theta|$: $|\mathscr{C}_{**}(X)| \to |BJ_G(X)_{**}|$ *is a homotopy equivalence.*

Proof. In any bisimplicial set X_{**}, the bisimplices of bidimension $(m, *)$, $0 \le * < \infty$, form a simplicial set X_{m*}. In [1] it is shown that a map of bisimplicial sets f_{**}: $X_{**} = Y_{**}$ induces a homotopy equivalence $|f_{**}|$: $|X_{**}| \to |Y_{**}|$ if each $|f_{m*}|$ is a homotopy equivalence. In our case, $X_{m,*}$ is a simplicial set $X_m \underset{G}{\times} W(G)$, and $Y_{m,*}$ is the simplicial set $BJ_G(X_m)$. Theorem 5 shows that $|\Theta_{m*}|$ is a homotopy equivalence. ■

PROPOSITION 7. $|\mathscr{C}_{**}(X)|$ *has the homotopy type of* $|X \underset{G}{\times} W(G)|/|X|$

Proof. $X \underset{G}{\times} W(G)/i(X)$ is the diagonal simplicial set associated to $C_{**}(X)$. In [1], it is shown that the realization of the diagonal simplicial set associated to X_{**} is homotopy equivalent to $|X_{**}|$. ■

PROPOSITION 8. $\Omega|BJ_G(X)_{**}| \cong |J_G(X)|$.

Proof. Let Γ_{**} be the bisimplicial group with $\Gamma_{mn} = J_G(X_m)$. Then if $\mathscr{B}_{**}$ is the bisimplicial set with $\mathscr{B}_{m,n} = W(J_G(X_m))_n$, the bisimplicial group Γ_{**} acts freely and bisimplicially on $\mathscr{B}_{**}$, so $|\Gamma_{**}|$ acts freely on $|\mathscr{B}_{**}|$. Moreover, since each $|\mathscr{B}_{m,*}|$ is contractible, [1] assures us that $|\mathscr{B}_{**}|$ is contractible. Therefore, $|\mathscr{B}_{**}|/|\Gamma_{**}|$ is a classifying space for the topological group $|\Gamma_{**}|$, so $\Omega(|\mathscr{B}_{**}|/|\Gamma_{**}|) \cong |\Gamma_{**}|$. But, $|\mathscr{B}_{**}|/|\Gamma_{**}| \cong |\mathscr{B}_{**}/\Gamma_{**}|$, and $\mathscr{B}_{**}/\Gamma_{**} \cong BJ_G(X)_{**}$, so $\Delta|BJ_G(X)_{**}| \cong |\Gamma_{**}|$. Finally, $|\Gamma_{**}| \simeq |J_G(X)|$, since $J_G(X)$ is the diagonal simplicial group associated to Γ_{**}.

THEOREM 9. *The topological group* $|J_G(X)|$ *has the homotopy type of* $\Omega|C(X)|$, *where* $C(X)$ *is the mapping cone on the inclusion* $X \to X \underset{G}{\times} \bar{W}(G)$.

Proof. Apply Ω to the equivalence $|\Theta|$ of Proposition 6. ■

Remark. The corresponding result for based G-sets and $J_G(X, *)$ is that $|J_G(X, *)|$ has the homotopy type of $\Omega\bar{C}(X)$, where $\bar{C}(X)$ is the mapping cone on the inclusion $X \to X \underset{G}{\times} W(G)$.

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